Int J Theor Phys (2008) 47: 2722–2739
DOI 10.1007/s10773-008-9711-4

A Further Generalized Lagrangian Density and its Special Cases

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Received: 7 December 2007 / Accepted: 13 March 2008 / Published online: 22 March 2008

Abstract By summarizing and extending the Lagrangian densities of general relativity and the Kibble's gauge theory of gravitation,a further generalized Lagrangian density for a gravitational system is obtained and analyzed in greater detail, which will extend the studying range for the theory of gravitation. Many special cases can be derived from this generalized Lagrangian density, and the general characteristics and some peculiarities of them will be described and discussed.

Keywords Lagrangian density · Couplings between fields · Conservation laws · Energy-momentum tensor density · Spin density

1 Introduction

In the theory of special relativity, the Lagrangian of matter field $\psi(x)$can be denoted by the functional form:

$$L_M(x) = L_M[\psi(x); \psi_{,\mu}(x)] \tag{1}$$

where $\psi_{,\mu}(x)$ is the ordinary derivative of $\psi(x)$. It is well known that, in the relativistic theories of gravitation, the Lagrangian density of matter field must be denoted by the functional form [1, 2]:

$$\sqrt{-g}L_M(x) = \sqrt{-g}L_M[\psi(x); \psi_{|\mu}(x); h^i_{.\mu}(x)] \tag{2}$$

where $h^i_{.\mu}(x)$ is the tetrad field, and $\psi_{|\mu}(x)$ is the covariant derivative of $\psi(x)$:

$$\psi_{|\mu}(x) = \psi_{,\mu}(x) + \frac{1}{2}\Gamma^{ij}_{..\mu}(x)\sigma_{ij}\psi(x) \tag{3}$$

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For the Kibble's gauge theory of gravitation [1], the frame connection $\Gamma^{ij}_{..\mu}(x)$ is independent field variables and the torsion must appear in the space-time. In this case the (2) can be generalized as

$$\sqrt{-g(x)}L_M(x) = \sqrt{-g(x)}L_M[\psi(x);\, \psi_{,\lambda}(x);\, h^i_{.\mu}(x);\, \Gamma^{ij}_{..\mu}(x)] \tag{4}$$

For the relativistic theories of gravitation in the space-time without torsion, $\Gamma^{ij}_{..\mu}(x)$ should not be independent field variables; and it will be proven in the Appendix that

$$\Gamma^{ij}_{..\mu} = \frac{1}{2}\eta^{jk}h_k^{;\nu}(h^i_{.\mu,\nu} - h^i_{.\nu,\mu}) + \frac{1}{2}\eta^{id}h_d^{;\sigma}(h^j_{.\sigma,\mu} - h^j_{.\mu,\sigma}) + \frac{1}{2}\eta^{jk}h_k^{;\nu}\eta^{id}h_d^{;\sigma}\eta_{ab}h^b_{.\mu}(h^a_{.\sigma,\nu} - h^a_{.\nu,\sigma}) \tag{5}$$

In this case (2) can be expressed as

$$\sqrt{-g(x)}L_M(x) = \sqrt{-g(x)}L_M[\psi(x);\, \psi_{,\lambda}(x);\, h^i_{.\mu}(x);\, h^i_{.\mu,\nu}(x)] \tag{6}$$

Since the majority of the fundamental matter fields are spinors, it is necessary to use tetrad field $h^i_{.\mu}(x)$ [2]. The metric field $g_{\mu\nu}(x)$ is expressed as $g_{\mu\nu}(x) = h^i_{.\mu}(x)h^j_{.\nu}(x)\eta_{ij}$, and we have $h_i^{;\mu}(x) = \eta_{ij}g^{\mu\nu}(x)h^j_{.\nu}(x)$; $h_{i\nu,\lambda}(x) = \frac{\partial}{\partial x^\lambda}h_{i\nu}(x)$; etc.

In the relativistic theories of gravitation,

$$\sqrt{-g}L_G(x) = \frac{\sqrt{-g}}{16\pi G}R(x) \tag{7}$$

is always adopted as the Lagrangian density of gravitational field [1, 2], where R is the scalar curvature. For the Kibble's gauge theory of gravitation [1], (7) can be generalized as

$$\sqrt{-g(x)}L_G(x) = \sqrt{-g(x)}L_G[h^i_{.\mu}(x);\, \Gamma^{ij}_{..\mu}(x);\, \Gamma^{ij}_{..\mu,\lambda}(x)] \tag{8}$$

For the relativistic theories of gravitation in the space-time without torsion (e.g. general relativity), after using (5), (7) can be expressed as [3]

$$\sqrt{-g(x)}L_G(x) = \sqrt{-g(x)}L_G[h^i_{.\mu}(x);\, h^i_{.\mu,\lambda}(x);\, h^i_{.\mu,\lambda\sigma}(x)] \tag{9}$$

In this paper,in order to conduct more detailed study on the general characteristics and the peculiarities of Lagrangian densities for some relativistic theories of gravitation, (4), (6) will be extended to the following expression:

$$\sqrt{-g(x)}L_M(x) = \sqrt{-g(x)}L_M[\psi(x);\, \psi_{,\lambda}(x);\, h^i_{.\mu}(x);\, h^i_{.\mu,\lambda}(x);\, \Gamma^{ij}_{..\mu}(x);\, \Gamma^{ij}_{..\mu,\lambda}(x)] \tag{10}$$

and (8) will be extended to the following expression:

$$\sqrt{-g(x)}L_G(x) = \sqrt{-g(x)}L_G[h^i_{.\mu}(x);\, h^i_{.\mu,\lambda}(x);\, \Gamma^{ij}_{..\mu}(x);\, \Gamma^{ij}_{..\mu,\lambda}(x)] \tag{11}$$

We will name $\sqrt{-g(x)}L(x) = \sqrt{-g(x)}L_M(x) + \sqrt{-g(x)}L_G(x)$ (where $\sqrt{-g(x)}L_M(x)$ and $\sqrt{-(g)}L_G(x)$ are denoted by (10) and (11) respectively) as a further generalized Lagrangian density. $\psi(x)$ represents the matter field, and $h^i_{.\mu}(x)$, $\Gamma^{ij}_{..\mu}(x)$ represent the gravitational fields. This further generalized Lagrangian density is significantly more general than the Lagrangian densities denoted by (4, 6) and (8, 9).

It must be indicated that, apart from describing a gravitational system with torsion, this further generalized Lagrangian density (*i.e.* (10, 11)) can be used also to describe a gravitational system without torsion. If (10, 11) are used to describe a gravitational system without torsion, it must be noted that $\Gamma^{ij}_{..\mu}(x)$ is function of $h^{i}_{.\mu}(x), h^{i}_{.\mu,\lambda}(x)$, and $\Gamma^{ij}_{..\mu,\lambda}(x)$ is function of $h^{i}_{.\mu}(x), h^{i}_{.\mu,\lambda}(x), h^{i}_{.\mu,\lambda\sigma}(x)$. So the (10) can be expressed as

$$\begin{aligned}\sqrt{-g(x)}L_M(x) &= \sqrt{-g(x)}L_M[\psi(x);\psi_{,\lambda}(x);h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x);\Gamma^{ab}_{..\alpha}[h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x)];\\ &\qquad \Gamma^{ab}_{..\alpha,\beta}[h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x);h^{i}_{.\mu,\lambda\sigma}(x)]]\\ &= \sqrt{-g(x)}L^{*}_M[\psi(x);\psi_{,\lambda}(x);h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x);h^{i}_{.\mu,\lambda\sigma}]\end{aligned} \tag{12}$$

and the (11) can be expressed as

$$\begin{aligned}\sqrt{-g(x)}L_G(x) &= \sqrt{-g(x)}L_G[h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x);\Gamma^{ab}_{..\alpha}[h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x)];\\ &\qquad \Gamma^{ab}_{..\alpha,\beta}[h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x);h^{i}_{.\mu,\lambda\sigma}]]\\ &= \sqrt{-g(x)}L^{*}_G[h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x);h^{i}_{.\mu,\lambda\sigma}(x)]\end{aligned} \tag{13}$$

For the relativistic theories of gravitation in the space-time with torsion, besides (4), the following Lagrangian densities

$$\sqrt{-g(x)}L_M(x) = \sqrt{-g(x)}L_M[\psi(x);\psi_{,\lambda}(x);h^{i}_{.\mu}(x);h^{i}_{.\mu,\lambda}(x);\Gamma^{ij}_{..\mu}(x)] \tag{14}$$

$$\sqrt{-g(x)}L_M(x) = \sqrt{-g(x)}L_M[\psi(x);\psi_{,\lambda}(x);h^{i}_{.\mu}(x);\Gamma^{ij}_{..\mu}(x);\Gamma^{ij}_{..\mu,\lambda}(x)] \tag{15}$$

are also the special cases of (10). By means of studying the further generalized Lagrangian density and its special cases, their general characteristics and peculiarity can be shown clearly.

The further generalized Lagrangian density summarizes many properties of various theories of gravitation. Below we shall prove that, (10) and (11) can be rewritten as

$$\begin{aligned}\sqrt{-g(x)}L_M(x) &= \sqrt{-g(x)}L^{\#}_M(x)\\ &= \sqrt{-g(x)}L^{\#}_M[\psi(x);\psi_{|\mu}(x);R^{ij}_{..\mu\nu}(x);T^{i}_{.\mu\nu}(x);h^{i}_{.\mu}(x)]\end{aligned} \tag{16}$$

and

$$\sqrt{-g(x)}L_G(x) = \sqrt{-g(x)}L^{\#}_G(x) = \sqrt{-g(x)}L^{\#}_G[R^{ij}_{..\mu\nu}(x);T^{i}_{.\mu\nu}(x);h^{i}_{.\mu}(x)] \tag{17}$$

$R^{ij}_{..\mu\nu}$ is the curvature tensor with mixed indices,

$$R^{ij}_{..\mu\nu} = \Gamma^{ij}_{..\nu,\mu} - \Gamma^{ij}_{..\mu,\nu} + \delta^{i}_{m}\eta_{nk}(\Gamma^{mn}_{..\mu}\Gamma^{kj}_{..\nu} - \Gamma^{mn}_{..\nu}\Gamma^{kj}_{..\mu}) \tag{18}$$

$T^{i}_{.\mu\nu}$ is the torsion tensor with mixed indices,

$$T^{i}_{.\mu\nu} = \frac{1}{2}\{h^{i}_{.\mu,\nu} - h^{i}_{.\nu,\mu} + \delta^{i}_{m}\eta_{nk}(\Gamma^{mn}_{..\nu}h^{k}_{.\mu} - \Gamma^{mn}_{..\mu}h^{k}_{.\nu})\} \tag{19}$$

The physical meaning of (16) is that the gravitational fields could act on the matter field only through covariant derivative, curvature of space-time, and torsion of space-time.

Therefore the forms of couplings between the gravitational fields and matter field might be $\Gamma^{ij}_{..\mu}\sigma_{ij}\psi$, $R|\psi|^2$, $R^{ij}_{..\mu\nu}R^{..\mu\nu}_{ij}|\psi|^2$ or $T^{i}_{.\mu\nu}T^{.\mu\nu}_{i}|\psi|^2$, etc. The coupling $\Gamma^{ij}_{..\mu}\sigma_{ij}\psi$ contained in the covariant derivative $\psi_{|\mu}(x)$ is called the minimal coupling, which is well known in the general relativity and the gauge theory of gravitation. Equation (16) tells us that in addition to the minimal coupling, there might be other complicated couplings in theory. But the general opinions of physicists are that the effects of those couplings composed of curvature or torsion are very small and could be neglected always.

The physical meaning of (17) is that the Lagrangian of gravitational field is composed of curvature tensor field and torsion tensor field. Because $L^{\#}_{G}(x)$ is both a coordinate scalar and a frame scalar, the possible terms involved in $L^{\#}_{G}(x)$ are scalars constructed from $R^{ij}_{..\mu\nu}(x)$, $T^{i}_{.\mu\nu}(x)$, $h^{i}_{.\mu}(x)$. For example: $L^{\#}_{G}(x) = aR + bR^{ij}_{..\mu\nu}R^{..\mu\nu}_{ij} + cT^{i}_{.\mu\nu}T^{.\mu\nu}_{i}$, etc.; Obukhov and other have studied this type of $L^{\#}_{G}(x)$ [4]. The gravitational theories with these $L^{\#}_{G}(x)$ are called higher order gravitational theory, owing to the order of corresponding partial differential equations are greater than 2. The higher order gravitational theory might have some applications in astronomical physics [5].

If (17) is used to describe a gravitational system without torsion, then $T^{i}_{.\mu\nu}(x) = 0$ and $\overset{(\Gamma)}{R}{}^{\sigma}_{.\lambda\mu\nu} = \overset{(\{\})}{R}{}^{\sigma}_{.\lambda\mu\nu}$, as shown in the Appendix; the possible terms involved in $L_G(x)$ are only the scalar curvature $\overset{\{\}}{R} = h^{.\mu}_{i}h^{.j}_{j}R^{ij}_{..\mu\nu}$ and its power such as $\overset{\{\}}{R}{}^2\ldots$. As a special case, $L_G(x) = \frac{1}{16\pi G}[R(x)]$ is chosen in general relativity. Although (9) tells us that there are terms $h^{i}_{.\mu,\lambda\sigma}(x)$ outwardly in $L_G(x) = \frac{1}{16\pi G}[R(x)]$, yet the Einstein field equations are still second order partial differential equations, because $\delta\int\sqrt{-g(x)}Rd^4x = \int\sqrt{-g(x)}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}d^4x$ [6].

To speak in general, the further generalized Lagrangian density and its special cases have direct bearings with the spacetime torsion and the gauge theory of gravitation. Hehl and others have studied the spacetime torsion and the gauge theory of gravitation for a long time, the references [7, 8] are the summations of their works.

The major objective of this paper is to study the general characteristics and some peculiarities of the further generalized Lagrangian density. We shall suppose that the spacetime has vanishing nonmetricity [8] and the gravitational system has the fundamental symmetry explained in Sect. 2.

2 The Symmetry of the Lagrangian Densities for a Gravitational System

Symmetries exist universally in physical systems. We suppose that one fundamental symmetry of a gravitational system is that the action integrals

$$I_M = \int\sqrt{-g(x)}L_M(x)d^4x, \qquad I_G = \int\sqrt{-g(x)}L_G(x)d^4x \quad \text{and}$$

$$I = I_M + I_G = \int\sqrt{-g(x)}(L_M(x) + L_G(x))d^4x$$

satisfy $\delta I_M = 0$, $\delta I_G = 0$ and $\delta I = 0$ respectively under the following two simultaneous transformations [1, 7]:

(1) the infinitesimal general coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \tag{20}$$

(2) the local Lorentz transformation of tetrad frame

$$e_i(x) \to e'_i(x') = e_i(x) - \varepsilon^{mn}(x)\delta^j_m \eta_{ni} e_j(x) \tag{21}$$

The sufficient condition of an action integral $I = \int \sqrt{-g(x)} L(x) d^4x$ being $\delta I = 0$ under above transformations is [1, 9]:

$$\delta_0(\sqrt{-g}L) + (\xi^\mu \sqrt{-g}L)_{,\mu} \equiv 0 \tag{22}$$

where δ_0 represents the variation at a fixed value of x. For the most generalized Lagrangian density we have

$$\begin{aligned}\delta_0(\sqrt{-g}L_M) = {} & \frac{\partial(\sqrt{-g}L_M)}{\partial\psi}\delta_0\psi + \frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\delta_0\psi_{,\lambda} + \frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu} \\ & + \frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}}\delta_0 h^i_{.\mu,\lambda} + \frac{\partial(\sqrt{-g}L_M)}{\partial \Gamma^{ij}_{..\mu}}\delta_0\Gamma^{ij}_{..\mu} \\ & + \frac{\partial(\sqrt{-g}L_M)}{\partial \Gamma^{ij}_{..\mu,\lambda}}\delta_0\Gamma^{ij}_{..\mu,\lambda}\end{aligned} \tag{23}$$

$$\begin{aligned}\delta_0(\sqrt{-g}L_G) = {} & \frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu} + \frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu,\lambda}}\delta_0 h^i_{.\mu,\lambda} \\ & + \frac{\partial(\sqrt{-g}L_G)}{\partial \Gamma^{ij}_{..\mu}}\delta_0\Gamma^{ij}_{..\mu} + \frac{\partial(\sqrt{-g}L_G)}{\partial \Gamma^{ij}_{..\mu,\lambda}}\delta_0\Gamma^{ij}_{..\mu,\lambda}\end{aligned} \tag{24}$$

Let $\Lambda = L_M$ or $\Lambda = L_G$ or $\Lambda = L_M + L_G$, because of the independent arbitrariness of $\varepsilon^{mn}_{,}(x)$, $\varepsilon^{mn}_{,\lambda}(x)$, $\varepsilon^{mn}_{,\lambda\sigma}(x)$, $\xi^\alpha(x)$, $\xi^\alpha_{,\mu}(x)$ and $\xi^\alpha_{,\mu\lambda}(x)$, It is not difficult to derive the following identities [10]:

$$\begin{aligned}&\frac{1}{2}\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi}\sigma_{mn}\psi + \frac{1}{2}\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi_{,\lambda} + \frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu}}h_{n\mu} \\ &\quad + \frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu,\lambda}}h_{n\mu,\lambda} + 2\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{km}_{..\mu}}\Gamma^k_{.n\mu} + 2\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu,\lambda} = 0\end{aligned} \tag{25}$$

$$\frac{1}{2}\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi + \frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu,\lambda}}h_{n\mu} + 2\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu} = \frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{mn}_{..\lambda}} \tag{26}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{mn}_{..\mu,\nu}} = -\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{mn}_{..\nu,\mu}} \tag{27}$$

$$\begin{aligned}&\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi}\psi_{,\alpha} + \frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\psi_{,\lambda\alpha} + \frac{\partial(\sqrt{-g}\Lambda)}{\partial h^i_{.\mu}}h^i_{.\mu,\alpha} \\ &\quad + \frac{\partial(\sqrt{-g}\Lambda)}{\partial h^i_{.\mu,\lambda}}h^i_{.\mu,\alpha\lambda} + \frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ij}_{..\mu}}\Gamma^{ij}_{..\mu,\alpha} \\ &\quad + \frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha\lambda} - (\sqrt{-g}\Lambda)_{,\alpha} = 0\end{aligned} \tag{28}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\psi_{,\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^i_{.\lambda}}h^i_{\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^i_{.\mu,\lambda}}(h^i_{.\mu,\alpha}-h^i_{.\alpha,\mu})$$
$$+\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ij}_{..\lambda}}\Gamma^{ij}_{..\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ij}_{..\mu,\lambda}}(\Gamma^{ij}_{..\mu,\alpha}-\Gamma^{ij}_{..\alpha,\mu})-\sqrt{-g}\Lambda\delta^{\lambda}_{\alpha}=0 \tag{29}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^i_{.\mu,\lambda}}h^i_{.\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\alpha}=0 \tag{30}$$

From (30) and (27), it is found that there must exist another identity:

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^i_{.\mu,\nu}}=-\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^i_{.\nu,\mu}} \tag{31}$$

It will be shown bellow that many properties of a gravitational system can be derived from the above identities. If $\Lambda=L_G$, then $\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi}=0$, $\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}=0$; the terms $\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi}$ and $\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}$ in (25, 26, 28, 29) will be absent.

When (10, 11) are used to describe a gravitational system without torsion, from (12) we have

$$\begin{aligned}
\delta_0(\sqrt{-g}L_M)&=\frac{\partial(\sqrt{-g}L_M)}{\partial\psi}\delta_0\psi+\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\delta_0\psi_{,\lambda}+\left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu}}\right)_{\Gamma}\delta_0 h^i_{.\mu}\\
&\quad+\left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}}\right)_{\Gamma}\delta_0 h^i_{.\mu,\lambda}+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu}\\
&\quad+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu,\lambda}}\delta_0 h^i_{.\mu,\lambda}+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu}\\
&\quad+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda}}\delta_0 h^i_{.\mu,\lambda}+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda\sigma}}\delta_0 h^i_{.\mu,\lambda\sigma}\\
&=\frac{\partial(\sqrt{-g}L^*_M)}{\partial\psi}\delta_0\psi+\frac{\partial(\sqrt{-g}L^*_M)}{\partial\psi_{,\lambda}}\delta_0\psi_{,\lambda}+\frac{\partial(\sqrt{-g}L^*_M)}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu}\\
&\quad+\frac{\partial(\sqrt{-g}L^*_M)}{\partial h^i_{.\mu,\lambda}}\delta_0 h^i_{.\mu,\lambda}+\frac{\partial(\sqrt{-g}L^*_M)}{\partial h^i_{.\mu,\lambda\sigma}}\delta_0 h^i_{.\mu,\lambda\sigma}
\end{aligned} \tag{32}$$

where $(\frac{\partial}{\partial})_{\Gamma}$ denote the partial derivative at the constant values of $\Gamma^{ab}_{..\alpha}(x)$ and $\Gamma^{ab}_{..\alpha,\beta}(x)$. Hence we get

$$\frac{\partial(\sqrt{-g}L^*_M)}{\partial\psi}=\frac{\partial(\sqrt{-g}L_M)}{\partial\psi} \tag{33}$$

$$\frac{\partial(\sqrt{-g}L^*_M)}{\partial\psi_{,\lambda}}=\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}} \tag{34}$$

$$\frac{\partial(\sqrt{-g}L^*_M)}{\partial h^i_{.\mu}}=\left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu}}\right)_{\Gamma}+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu}}+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu}} \tag{35}$$

$$\frac{\partial(\sqrt{-g}L_M^*)}{\partial h^i_{.\mu,\lambda}}=\left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}}\right)_\Gamma+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu,\lambda}}+\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda}} \tag{36}$$

$$\frac{\partial(\sqrt{-g}L_M^*)}{\partial h^i_{.\mu,\lambda\sigma}}=\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda\sigma}} \tag{37}$$

From (13) we have

$$\begin{aligned}\delta_0(\sqrt{-g}L_G)&=\left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu}}\right)_\Gamma\delta_0h^i_{.\mu}+\left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu,\lambda}}\right)_\Gamma\delta_0h^i_{.\mu,\lambda}\\&\quad+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu}}\delta_0h^i_{.\mu}+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu,\lambda}}\delta_0h^i_{.\mu,\lambda}\\&\quad+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu}}\delta_0h^i_{.\mu}+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda}}\delta_0h^i_{.\mu,\lambda}\\&\quad+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda\sigma}}\delta_0h^i_{.\mu,\lambda\sigma}\\&=\frac{\partial(\sqrt{-g}L_G^*)}{\partial h^i_{.\mu}}\delta_0h^i_{.\mu}+\frac{\partial(\sqrt{-g}L_G^*)}{\partial h^i_{.\mu,\lambda}}\delta_0h^i_{.\mu,\lambda}+\frac{\partial(\sqrt{-g}L_G^*)}{\partial h^i_{.\mu,\lambda\sigma}}\delta_0h^i_{.\mu,\lambda\sigma}\end{aligned} \tag{38}$$

Hence we get

$$\frac{\partial(\sqrt{-g}L_G^*)}{\partial h^i_{.\mu}}=\left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu}}\right)_\Gamma+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu}}+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu}} \tag{39}$$

$$\frac{\partial(\sqrt{-g}L_G^*)}{\partial h^i_{.\mu,\lambda}}=\left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu,\lambda}}\right)_\Gamma+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha}}\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^i_{.\mu,\lambda}}+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda}} \tag{40}$$

$$\frac{\partial(\sqrt{-g}L_G^*)}{\partial h^i_{.\mu,\lambda\sigma}}=\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^i_{.\mu,\lambda\sigma}} \tag{41}$$

On the other hand it is evident that L_M^* and L_G^* relating to (12, 13) satisfy also $\delta_0(\sqrt{-g}\Lambda)+(\xi^\mu\sqrt{-g}\Lambda)_{,\mu}\equiv 0$, where $\Lambda=L_M^*$ or $\Lambda=L_G^*$ or $\Lambda=L_M^*+L_G^*$. Owing to the independent arbitrariness of $\varepsilon^{mn}_{,}(x)$, $\varepsilon^{mn}_{,\lambda}(x)$, $\varepsilon^{mn}_{,\lambda\sigma}(x)$, $\xi^\alpha(x)$, $\xi^\alpha_{,\mu}(x)$, $\xi^\alpha_{,\mu\lambda}(x)$ and $\xi^\alpha_{,\mu\lambda\sigma}(x)$, we obtain another set of identities [3] (if $\Lambda=L_G^*$, then $\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi}=0$, $\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}=0$):

$$\begin{aligned}&\frac{1}{2}\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi}\sigma_{mn}\psi+\frac{1}{2}\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi_{,\lambda}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu}}h_{n\mu}\\&\quad+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu,\lambda}}h_{n\mu,\lambda}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu,\lambda\sigma}}h_{n\mu,\lambda\sigma}=0\end{aligned} \tag{42}$$

$$\frac{1}{2}\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu,\lambda}}h_{n\mu}+2\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu,\lambda\sigma}}h_{n\mu,\sigma}=0 \tag{43}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^m_{.\mu,\lambda\sigma}}h_{n\mu}=\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^n_{.\mu,\lambda\sigma}}h_{m\mu} \tag{44}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi}\psi_{,\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\psi_{,\lambda\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\mu}}h^{i}_{.\mu,\alpha}$$

$$+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\mu,\lambda}}h^{i}_{.\mu,\lambda\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\mu,\lambda\sigma}}h^{i}_{.\mu,\lambda\sigma\alpha}-(\sqrt{-g}\,\Lambda)_{,\alpha}=0 \tag{45}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial\psi_{,\lambda}}\psi_{,\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\lambda}}h^{i}_{.\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\mu,\lambda}}h^{i}_{.\mu,\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\lambda,\mu}}h^{i}_{.\alpha,\mu}$$

$$+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\lambda,\mu\sigma}}h^{i}_{.\alpha,\mu\sigma}+2\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\mu,\lambda\sigma}}h^{i}_{.\mu,\sigma\alpha}-\sqrt{-g}\Lambda\delta^{\lambda}_{\alpha}=0 \tag{46}$$

$$\frac{\partial(\sqrt{-g}\,\Lambda)}{\partial h^{i}_{.\mu,\lambda}}h^{i}_{.\alpha}+\frac{\partial(\sqrt{-g}\,\Lambda)}{\partial h^{i}_{.\mu,\lambda\sigma}}h^{i}_{.\alpha,\sigma}+\frac{\partial(\sqrt{-g}\,\Lambda)}{\partial h^{i}_{.\sigma,\lambda\mu}}h^{i}_{.\sigma,\alpha}-\frac{\partial}{\partial x^{\sigma}}\left(\frac{\partial(\sqrt{-g}\,\Lambda)}{\partial h^{i}_{.\mu,\lambda\sigma}}\right)h^{i}_{.\alpha}$$

$$=-\frac{\partial}{\partial x^{\sigma}}\left(\frac{\partial(\sqrt{-g}\,\Lambda)}{\partial h^{i}_{.\mu,\lambda\sigma}}h^{i}_{.\alpha}\right) \tag{47}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\mu,\lambda\sigma}}h^{i}_{.\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\lambda,\sigma\mu}}h^{i}_{.\alpha}+\frac{\partial(\sqrt{-g}\Lambda)}{\partial h^{i}_{.\sigma,\mu\lambda}}h^{i}_{.\alpha}=0 \tag{48}$$

In addition there are the relations:

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ab}_{..\alpha}}\delta_0\Gamma^{ab}_{..\alpha}=\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ab}_{..\alpha}}\left(\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^{i}_{.\mu}}\delta_0 h^{i}_{.\mu}+\frac{\partial\Gamma^{ab}_{..\alpha}}{\partial h^{i}_{.\mu,\lambda}}\delta_0 h^{i}_{.\mu,\lambda}\right) \tag{49}$$

$$\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\delta_0\Gamma^{ab}_{..\alpha,\beta}=\frac{\partial(\sqrt{-g}\Lambda)}{\partial\Gamma^{ab}_{..\alpha,\beta}}\left(\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^{i}_{.\mu}}\delta_0 h^{i}_{.\mu}+\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^{i}_{.\mu,\lambda}}\delta_0 h^{i}_{.\mu,\lambda}\right.$$

$$\left.+\frac{\partial\Gamma^{ab}_{..\alpha,\beta}}{\partial h^{i}_{.\mu,\lambda\sigma}}\delta_0 h^{i}_{.\mu,\lambda\sigma}\right) \tag{50}$$

For the case without torsion, utilizing these relations and those in (33–37, 39–41), and carrying out some complicated calculations, it can be proven that the identities (25–30) are equivalent to the identities (42–48).

3 Possible Forms of the Lagrangians under the Symmetry of Transformations (20, 21)

In this section we will prove that, due to the requirement of the action integrals of a gravitational system being invariant under the transformations (20, 21), the possible forms of the Lagrangian densities (10) and (11) might be:

$$\sqrt{-g(x)}L_M(x)=\sqrt{-g(x)}L^{\#}_M[\psi(x);\,\psi_{|\mu}(x);\,R^{ij}_{..\mu\nu}(x);\,T^{i}_{.\mu\nu}(x);\,h^{i}_{.\mu}(x)] \tag{16}$$

and

$$\sqrt{-g}L_G(x)=\sqrt{-g}L^{\#}_G[R^{ij}_{..\mu\nu}(x);\,T^{i}_{.\mu\nu}(x);\,h^{i}_{.\mu}(x)] \tag{17}$$

respectively. The proof of (16) is given in the following:

Equation (27) means that $\Gamma^{ij}_{..\mu,\nu}(x)$ appears in $\sqrt{-g}L_M(x)$ only through a curvature tensor field $R^{ij}_{..\mu\nu}(x)$ because $2\,\frac{\partial(\sqrt{-g}L_M)}{\partial R^{ij}_{..\nu\mu}}\equiv\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu,\nu}}$; (31) means that $h^{i}_{.\mu,\nu}(x)$ appears in

$\sqrt{-g}L_M(x)$ only through torsion tensor field $T^i_{.\mu\nu}(x)$ because $\frac{\partial(\sqrt{-g}L_M}{\partial T^i_{.\mu\nu}} \equiv \frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\nu}}$; (26) means that $\Gamma^{mn}_{..\nu}(x)$ appears in $\sqrt{-g}L_M(x)$ only through covariant derivative $\psi_{|\mu}(x)$ and curvature tensor field $R^{ij}_{..\mu\nu}(x)$ and torsion tensor field $T^i_{.\mu\nu}(x)$ because

$$\begin{aligned}\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{mn}_{..\lambda}} &= \frac{1}{2}\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi + \frac{\partial(\sqrt{-g}L_M)}{\partial h^m_{.\mu,\lambda}}h_{n\mu} + 2\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu}\\ &= \frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{|\mu}}\frac{\partial\psi_{|\mu}}{\partial\Gamma^{mn}_{..\lambda}} + \frac{\partial(\sqrt{-g}L_M)}{\partial T^i_{.\alpha\beta}}\frac{\partial T^i_{.\alpha\beta}}{\partial\Gamma^{mn}_{..\lambda}}\\ &\quad + \frac{\partial(\sqrt{-g}L_M)}{\partial R^{ij}_{..\alpha\beta}}\frac{\partial R^{ij}_{..\alpha\beta}}{\partial\Gamma^{mn}_{..\lambda}}\end{aligned}$$

Hence the matter Lagrangian density $\sqrt{-g}L_M(x)$ should take the form denoted by (16).

On the other hand if there exists the relation (16), we must have:

$$\begin{aligned}\sqrt{-g(x)}L_M(x) &= \sqrt{-g(x)}L^\#_M(x)\\ &= \sqrt{-g(x)}L^\#_M[\psi(x);\psi_{|\alpha}[\psi(x);\psi_{,\mu}(x);\Gamma^{ij}_{..\mu}(x)];\\ &\qquad R^{ab}_{..\alpha\beta}[\Gamma^{ij}_{..\mu}(x);\Gamma^{ij}_{..\mu,\lambda}(x)];T^a_{.\alpha\beta}[h^i_{.\mu}(x);h^i_{.\mu.\lambda}(x);\\ &\qquad \Gamma^{ij}_{..\mu}(x)];h^i_{.\mu}(x)]\\ &= \sqrt{-g(x)}L_M[\psi(x);\psi_{,\lambda}(x);h^i_{.\mu}(x);h^i_{.\mu,\lambda}(x);\Gamma^{ij}_{..\mu}(x);\Gamma^{ij}_{..\mu,\lambda}(x)]\end{aligned}\tag{51}$$

Therefore from (51)

$$\begin{aligned}\delta_0(\sqrt{-g}L^\#_M) &= \frac{\partial(\sqrt{-g}L^\#_M)}{\partial\psi}\delta_0\psi + \frac{\partial(\sqrt{-g}L^\#_M)}{\partial\psi_{|\alpha}}\left(\frac{\partial\psi_{|\alpha}}{\partial\psi}\delta_0\psi + \frac{\partial\psi_{|\alpha}}{\partial\psi_{,\mu}}\delta_0\psi_{,\mu} + \frac{\partial\psi_{|\alpha}}{\partial\Gamma^{ij}_{..\mu}}\delta_0\Gamma^{ij}_{..\mu}\right)\\ &\quad + \frac{\partial(\sqrt{-g}L^\#_M)}{\partial T^a_{.\alpha\beta}}\left(\frac{\partial T^a_{.\alpha\beta}}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu} + \frac{\partial T^a_{.\alpha\beta}}{\partial h^i_{.\mu,\lambda}}\delta_0 h^i_{.\mu,\lambda} + \frac{\partial T^a_{.\alpha\beta}}{\partial\Gamma^{ij}_{.\mu}}\delta_0\Gamma^{ij}_{.\mu}\right)\\ &\quad + \frac{\partial(\sqrt{-g}L^\#_M)}{\partial R^{ab}_{..\alpha\beta}}\left(\frac{\partial R^{ab}_{..\alpha\beta}}{\partial\Gamma^{ij}_{..\mu}}\delta_0\Gamma^{ij}_{..\mu} + \frac{\partial R^{ab}_{..\alpha\beta}}{\partial\Gamma^{ij}_{..\mu,\lambda}}\delta_0\Gamma^{ij}_{..\mu,\lambda}\right) + \frac{\partial(\sqrt{-g}L^\#_M)}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu}\\ &= \delta_0(\sqrt{-g}L_M) = \frac{\partial(\sqrt{-g}L_M)}{\partial\psi}\delta_0\psi + \frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\delta_0\psi_{,\lambda} + \frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu}\\ &\quad + \frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}}\delta_0 h^i_{.\mu,\lambda} + \frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu}}\delta_0\Gamma^{ij}_{..\mu} + \frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\delta_0\Gamma^{ij}_{..\mu,\lambda}\end{aligned}\tag{52}$$

Thus we have

$$\frac{\partial(\sqrt{-g}L_M)}{\partial\psi} = \frac{\partial(\sqrt{-g}L^\#_M)}{\partial\psi} + \frac{\partial(\sqrt{-g}L^\#_M)}{\partial\psi_{|\alpha}}\frac{\partial\psi_{|\alpha}}{\partial\psi}\tag{53}$$

$$\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}} = \frac{\partial(\sqrt{-g}L^\#_M)}{\partial\psi_{|\alpha}}\frac{\partial\psi_{|\alpha}}{\partial\psi_{,\lambda}}\tag{54}$$

$$\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu}} = \frac{\partial(\sqrt{-g}L^\#_M)}{\partial h^i_{.\mu}} + \frac{\partial(\sqrt{-g}L^\#_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial h^i_{.\mu}}\tag{55}$$

$$\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}} = \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial h^i_{.\mu,\lambda}} \tag{56}$$

$$\frac{\partial(\sqrt{-g}L_M)}{\partial \Gamma^{ij}_{..\mu}} = \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\alpha}}\frac{\partial \psi_{|\alpha}}{\partial \Gamma^{ij}_{..\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial \Gamma^{ij}_{..\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{ij}_{..\mu}} \tag{57}$$

$$\frac{\partial(\sqrt{-g}L_M)}{\partial \Gamma^{ij}_{..\mu,\lambda}} = \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{ij}_{..\mu,\lambda}} \tag{58}$$

Using (53–58) and (25–31) we have the following identities:

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\alpha}}\frac{\partial \psi_{|\alpha}}{\partial \psi}\right)\sigma_{mn}\psi + \frac{1}{2}\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\alpha}}\frac{\partial \psi_{|\alpha}}{\partial \psi_{,\lambda}}\sigma_{mn}\psi_{,\lambda}\\
&+\left(\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial h^m_{.\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial h^m_{.\mu}}\right)h_{n\mu} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial h^m_{.\mu,\lambda}}h_{n\mu,\lambda}\\
&+2\left(\frac{\partial\sqrt{-g}L^{\#}_M}{\partial \psi_{|\alpha}}\frac{\partial \psi_{|\alpha}}{\partial \Gamma^{km}_{..\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial \Gamma^{km}_{..\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{km}_{..\mu}}\right)\Gamma^k_{.n\mu}\\
&+2\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu,\lambda} = 0
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\alpha}}\frac{\partial \psi_{|\alpha}}{\partial \psi_{,\lambda}}\sigma_{mn}\psi + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial h^m_{.\mu,\lambda}}h_{n\mu} + 2\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu}\\
&= \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\alpha}}\frac{\partial \psi_{|\alpha}}{\partial \Gamma^{mn}_{..\lambda}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial \Gamma^{mn}_{..\lambda}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{mn}_{..\lambda}}
\end{aligned} \tag{60}$$

$$\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{mn}_{..\mu,\nu}} = -\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\alpha\beta}}\frac{\partial R^{ab}_{..\alpha\beta}}{\partial \Gamma^{mn}_{..\nu,\mu}} \tag{61}$$

$$\begin{aligned}
&\left(\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\beta}}\frac{\partial \psi_{|\beta}}{\partial \psi}\right)\psi_{,\alpha} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\beta}}\frac{\partial \psi_{|\beta}}{\partial \psi_{,\lambda}}\psi_{,\lambda\alpha}\\
&+\left(\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial h^i_{.\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\beta\sigma}}\frac{\partial T^a_{.\beta\sigma}}{\partial h^i_{.\mu}}\right)h^i_{.\mu,\alpha} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\beta\sigma}}\frac{\partial T^a_{.\beta\sigma}}{\partial h^i_{.\mu,\lambda}}h^i_{.\mu,\alpha\lambda}\\
&+\left(\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\beta}}\frac{\partial \psi_{|\beta}}{\partial \Gamma^{ij}_{..\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\beta\sigma}}\frac{\partial T^a_{.\beta\sigma}}{\partial \Gamma^{ij}_{..\mu}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\beta\sigma}}\frac{\partial R^{ab}_{..\beta\sigma}}{\partial \Gamma^{ij}_{..\mu}}\right)\Gamma^{ij}_{..\mu,\alpha}\\
&+\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial R^{ab}_{..\beta\sigma}}\frac{\partial R^{ab}_{..\beta\sigma}}{\partial \Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha\lambda} - (\sqrt{-g}L^{\#}_M)_{,\alpha} = 0
\end{aligned} \tag{62}$$

$$\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial \psi_{|\rho}}\frac{\partial \psi_{|\rho}}{\partial \psi_{,\lambda}}\psi_{,\alpha} + \left(\frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial h^i_{.\lambda}} + \frac{\partial(\sqrt{-g}L^{\#}_M)}{\partial T^a_{.\beta\sigma}}\frac{\partial T^a_{.\beta\sigma}}{\partial h^i_{.\lambda}}\right)h^i_{.\alpha}$$

$$+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial T^a_{.\beta\sigma}}\frac{\partial T^a_{.\beta\sigma}}{\partial h^i_{.\mu,\lambda}}(h^i_{.\mu,\alpha}-h^i_{.\alpha,\mu})$$
$$+\left(\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial \psi_{|\rho}}\frac{\partial \psi_{|\rho}}{\partial \Gamma^{ij}_{..\lambda}}+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial T^a_{.\beta\sigma}}\frac{\partial T^a_{.\beta\sigma}}{\partial \Gamma^{ij}_{..\lambda}}+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial R^{ab}_{..\beta\sigma}}\frac{\partial R^{ab}_{..\beta\sigma}}{\partial \Gamma^{ij}_{..\lambda}}\right)\Gamma^{ij}_{..\alpha}$$
$$+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial R^{ab}_{..\beta\sigma}}\frac{\partial R^{ab}_{..\beta\sigma}}{\partial \Gamma^{ij}_{..\mu,\lambda}}(\Gamma^{ij}_{..\mu,\alpha}-\Gamma^{ij}_{..\alpha,\mu})-\sqrt{-g}L_M^{\#}\delta^\lambda_\alpha=0 \tag{63}$$

$$\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial T^a_{.\beta\sigma}}\frac{\partial T^a_{.\beta\sigma}}{\partial h^i_{.\mu,\lambda}}h^i_{.\alpha}+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial R^{ab}_{..\beta\sigma}}\frac{\partial R^{ab}_{..\beta\sigma}}{\partial \Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\alpha}=0 \tag{64}$$

$$\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial h^i_{.\mu,\lambda}}=-\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial T^a_{.\alpha\beta}}\frac{\partial T^a_{.\alpha\beta}}{\partial h^i_{.\lambda,\mu}} \tag{65}$$

On the other hand, from (16) we also have:

$$\delta_0(\sqrt{-g}L_M^{\#})=\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial\psi}\delta_0\psi+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial\psi_{|\lambda}}\delta_0\psi_{|\lambda}+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial h^i_{.\mu}}\delta_0 h^i_{.\mu}$$
$$+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial R^{ab}_{..\alpha\beta}}\delta_0 R^{ab}_{..\alpha\beta}+\frac{\partial(\sqrt{-g}L_M^{\#})}{\partial T^a_{.\alpha\beta}}\delta_0 T^a_{.\alpha\beta} \tag{66}$$

where

$$\delta_0\psi_{|\lambda}=\delta_0\psi_{,\lambda}+\frac{1}{2}\Gamma^{mn}_{..\lambda}\sigma_{mn}\delta_0\psi+\frac{1}{2}(\delta_0\Gamma^{mn}_{..\lambda})\sigma_{mn}\psi \tag{67}$$

$$\delta_0 R^{ab}_{..\alpha\beta}=\delta_0\Gamma^{ab}_{..\beta,\alpha}-\delta_0\Gamma^{ab}_{..\alpha,\beta}+\delta^a_m\eta_{nk}((\delta_0\Gamma^{mn}_{..\alpha})\Gamma^{kb}_{..\beta}$$
$$+\Gamma^{mn}_{..\alpha}(\delta_0\Gamma^{kb}_{..\beta})-(\delta_0\Gamma^{mn}_{..\beta})\Gamma^{kb}_{..\alpha}-\Gamma^{mn}_{..\beta}(\delta_0\Gamma^{kb}_{..\alpha})) \tag{68}$$

$$\delta_0 T^a_{.\alpha\beta}=\frac{1}{2}\{\delta_0 h^a_{.\alpha,\beta}-\delta_0 h^a_{.\beta,\alpha}+\delta^a_m\eta_{nk}((\delta_0\Gamma^{mn}_{..\beta})h^k_{.\alpha}+\Gamma^{mn}_{..\beta}(\delta_0 h^k_{.\alpha})$$
$$-(\delta_0\Gamma^{mn}_{..\alpha})h^k_{.\beta}-\Gamma^{mn}_{..\alpha}(\delta_0 h^k_{.\beta}))\} \tag{69}$$

Substituting (67–69) into (66) and using

$$\delta_0(\sqrt{-g}L_M^{\#})+(\xi^\mu\sqrt{-g}L_M^{\#})_{,\mu}\equiv 0$$

because of the independent arbitrariness of $\varepsilon^{mn}(x), \varepsilon^{mn}_{,\lambda}(x)$, $\varepsilon^{mn}_{,\lambda\sigma}(x)$, $\xi^\alpha(x)$, $\xi^\alpha_{,\mu}(x)$ and $\xi^\alpha_{,\mu\lambda}(x)$, after some lengthy calculations we can obtain the identities (59–64) one by one. Hence the identities obtained directly from

$$\sqrt{-g}L_M^{\#}[\psi;\psi_{|\alpha};R^{ab}_{..\alpha\beta};T^a_{.\alpha\beta};h^i_{.\mu}]$$

are just the same as those derived from

$$\sqrt{-g}L_M^{\#}[\psi;\psi_{|\alpha}[\psi;\psi_{,\mu};\Gamma^{ij}_{..\mu}];R^{ab}_{..\alpha\beta}[\Gamma^{ij}_{..\mu};\Gamma^{ij}_{..\mu,\lambda}];T^a_{.\alpha\beta}[h^i_{.\mu};h^i_{.\mu.\lambda};\Gamma^{ij}_{..\mu}];h^i_{.\mu}]$$
$$=\sqrt{-g}L_M[\psi;\psi_{,\lambda};h^i_{.\mu};h^i_{.\mu,\lambda};\Gamma^{ij}_{..\mu};\Gamma^{ij}_{..\mu,\lambda}]$$

The above analysis prove that the relation

$$\sqrt{-g(x)}L_M(x) = \sqrt{-g(x)}L_M[\psi(x); \psi_{,\mu}(x); h^i_{.\mu}(x); h^i_{.\mu,\lambda}(x); \Gamma^{ij}_{..\mu}(x); \Gamma^{ij}_{..\mu\lambda}(x)]$$
$$= \sqrt{-g(x)}L^\#_M(x) = \sqrt{-g(x)}L^\#_M[\psi(x); \psi_{|\mu}(x); R^{ij}_{..\mu\nu}(x); T^i_{.\mu\nu}(x); h^i_{.\mu}(x)]$$

must exist.

With the same method we can also prove the following relations:

$$\sqrt{-g(x)}L_G(x) = \sqrt{-g(x)}L_G[h^i_{.\mu}(x); h^i_{.\mu,\lambda}(x); \Gamma^{ij}_{..\mu}(x); \Gamma^{ij}_{..\mu\lambda}(x)]$$
$$= \sqrt{-g(x)}L^\#_G(x) = \sqrt{-g(x)}L^\#_G[R^{ij}_{..\mu\nu}(x); T^i_{.\mu\nu}(x); h^i_{.\mu}(x)]$$

Previously we tried to prove (16) in Ref. [11]. However there appear to be some errors and misprints in that proof. In the above analysis we believe we have corrected these flaws.

4 Conservation Laws for a Gravitational System with our Further Generalized Lagrangian Density

Below we shall derive the conservation laws for a gravitational system with our further generalized Lagrangian density denoted by (10, 11) from the identities (25–30) and equations of fields:

$$\frac{\partial(\sqrt{-g}L_M)}{\partial\psi} - \frac{\partial}{\partial x^\mu}\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\mu}} = 0 \tag{70}$$

$$\frac{\delta(\sqrt{-g}L_G)}{\delta h^i_{.\mu}} = \frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu}} - \frac{\partial}{\partial x^\lambda}\frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu,\lambda}}$$
$$= -\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu}} + \frac{\partial}{\partial x^\lambda}\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}} = -\frac{\delta(\sqrt{-g}L_M)}{\delta h^i_{.\mu}} \tag{71}$$

$$\frac{\delta(\sqrt{-g}L_G)}{\delta\Gamma^{ij}_{..\mu}} = \frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\mu}} - \frac{\partial}{\partial x^\lambda}\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\mu,\lambda}}$$
$$= -\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu}} + \frac{\partial}{\partial x^\lambda}\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu,\lambda}} = -\frac{\delta(\sqrt{-g}L_M)}{\delta\Gamma^{ij}_{..\mu}} \tag{72}$$

From them we can get the following relations:

$$\frac{\partial}{\partial x^\lambda}\Bigg(\sqrt{-g}(L_M+L_G)\delta^\lambda_\alpha - \frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\psi_{,\alpha} - \frac{\partial(\sqrt{-g}(L_M+L_G))}{\partial h^i_{.\mu,\lambda}}h^i_{.\mu,\alpha}$$
$$-\frac{\partial(\sqrt{-g}(L_M+L_G))}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha}\Bigg) = 0 \tag{73}$$

$$\frac{\partial}{\partial x^\lambda}\Bigg(\frac{1}{2}\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi + \frac{\partial(\sqrt{-g}(L_M+L_G))}{\partial h^m_{.\mu,\lambda}}h_{n\mu}$$
$$+2\frac{\partial(\sqrt{-g}(L_M+L_G))}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu}\Bigg) = 0 \tag{74}$$

$$\sqrt{-g}L_M\delta^\lambda_\alpha - \frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\psi_{,\alpha} - \frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}}h^i_{.\mu,\alpha} - \frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha}$$
$$= \left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\lambda}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\lambda,\mu}}\right)\right)h^i_{.\alpha}$$
$$+\left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\lambda}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\lambda,\mu}}\right)\right)\Gamma^{ij}_{..\alpha} \quad (75)$$

$$\sqrt{-g}L_G\delta^\lambda_\alpha - \frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu,\lambda}}h^i_{.\mu,\alpha} - \frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha}$$
$$= \left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\lambda}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\lambda,\mu}}\right)\right)h^i_{.\alpha}$$
$$+\left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\lambda}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\lambda,\mu}}\right)\right)\Gamma^{ij}_{..\alpha} \quad (76)$$

$$\frac{1}{2}\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi + \frac{\partial(\sqrt{-g}L_M)}{\partial h^m_{.\mu,\lambda}}h_{n\mu} + 2\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu}$$
$$-\frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right) = \frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{mn}_{..\lambda}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right) \quad (77)$$

$$\frac{\partial(\sqrt{-g}L_G)}{\partial h^m_{.\mu,\lambda}}h_{n\mu} + 2\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu}$$
$$-\frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right) = \frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{mn}_{..\lambda}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right) \quad (78)$$

Equation (73) might be regarded as conservation laws of energy-momentum tensor density for the gravitational system:

$$\frac{\partial}{\partial x^\lambda}(\sqrt{-g}t^\lambda_{(M)\alpha} + \sqrt{-g}t^\lambda_{(G)\alpha}) = 0 \quad (79)$$

where

$$\sqrt{-g}t^\lambda_{(M)\alpha} = \sqrt{-g}L_M\delta^\lambda_\alpha - \frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\psi_{,\alpha}$$
$$- \frac{\partial(\sqrt{-g}L_M)}{\partial h^i_{.\mu,\lambda}}h^i_{.\mu,\alpha} - \frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha} \quad (80)$$

and

$$\sqrt{-g}t^\lambda_{(G)\alpha} = \sqrt{-g}L_G\delta^\lambda_\alpha - \frac{\partial(\sqrt{-g}L_G)}{\partial h^i_{.\mu,\lambda}}h^i_{.\mu,\alpha} - \frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha} \quad (81)$$

might be interpreted as the energy-momentum tensor density of matter field and of gravitational field respectively. But we must indicate that $\sqrt{-g}t^{\cdot\lambda}_{(M)\alpha}$ and $\sqrt{-g}t^{\cdot\lambda}_{(G)\alpha}$ are not tensor densities and (79) is not a covariant relation. However if we use (75) to define

$$\begin{aligned}\sqrt{-g}T^{\lambda}_{(M)\alpha} &= \left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^{i}_{.\lambda}} - \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial h^{i}_{.\lambda,\mu}}\right)\right)h^{i}_{.\alpha} \\ &= \sqrt{-g}L_M\delta^{\lambda}_{\alpha} - \frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\psi_{,\alpha} \\ &\quad - \frac{\partial(\sqrt{-g}L_M)}{\partial h^{i}_{.\mu,\lambda}}h^{i}_{.\mu,\alpha} - \frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha} \\ &\quad - \left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\lambda}} - \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{ij}_{..\lambda,\mu}}\right)\right)\Gamma^{ij}_{..\alpha}\end{aligned} \tag{82}$$

and use (76) to define

$$\begin{aligned}\sqrt{-g}T^{\lambda}_{(G)\alpha} &= \left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^{i}_{.\lambda}} - \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial h^{i}_{.\lambda,\mu}}\right)\right)h^{i}_{.\alpha} \\ &= \sqrt{-g}L_G\delta^{\lambda}_{\alpha} - \frac{\partial(\sqrt{-g}L_G)}{\partial h^{i}_{.\mu,\lambda}}h^{i}_{.\mu,\alpha} \\ &\quad - \frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\mu,\lambda}}\Gamma^{ij}_{..\mu,\alpha} - \left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\lambda}} - \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{ij}_{..\lambda,\mu}}\right)\right)\Gamma^{ij}_{..\alpha}\end{aligned} \tag{83}$$

then we get

$$\sqrt{-g}T^{\lambda}_{(M)\alpha} + \sqrt{-g}T^{\lambda}_{(G)\alpha} = 0 \tag{84}$$

$$\frac{\partial}{\partial x^{\lambda}}(\sqrt{-g}T^{\lambda}_{(M)\alpha} + \sqrt{-g}T^{\lambda}_{(G)\alpha}) = 0 \tag{85}$$

Here $\sqrt{-g}T^{.\lambda}_{(M)\alpha}$ and $\sqrt{-g}T^{.\lambda}_{(G)\alpha}$ are tensor densities and (85) is a covariant relation. Hence we will take (84, 85) to be the conservation laws of energy-momentum tensor density for the gravitational system with our further generalized Lagrangian densities. Historically, Einstein had proposed other conservation laws of energy-momentum tensor density for a gravitational system [12]:

$$\frac{\partial}{\partial x^{\lambda}}(\sqrt{-g}T^{\lambda}_{(M)\alpha} + \sqrt{-g}t^{\sim\lambda}_{(G)\alpha}) = 0 \tag{86}$$

where $\sqrt{-g}t^{\sim\lambda}_{(G)\alpha} = \sqrt{-g}T^{\lambda}_{(G)\alpha} - \frac{\partial}{\partial x^{\beta}}u^{\lambda\beta}_{(G)\alpha}$, $\frac{\partial}{\partial x^{\beta}}u^{\lambda\beta}_{(G)\alpha} = -\frac{\partial}{\partial x^{\beta}}u^{\beta\lambda}_{(G)\alpha}$.

The virtues and defects about (85) and (86) have been discussed thoroughly in Refs. [3, 12]. Because (84, 85) have more logical basis and rich physical contents, the author believes that the conservation laws in (84, 85) might be better than Einstein's conservation laws (86) [3, 12] and could be tested by future experiments and observations.

Equation (74) might be regarded as conservation laws of spin density for the gravitational system:

$$\frac{\partial}{\partial x^{\lambda}}(\sqrt{-g}s^{\lambda}_{(M)mn} + \sqrt{-g}s^{\lambda}_{(G)mn}) = 0 \tag{87}$$

where

$$\sqrt{-g}s^{\lambda}_{(M)mn}=\frac{1}{2}\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi+\frac{\partial(\sqrt{-g}L_M)}{\partial h^m_{.\mu,\lambda}}h_{n\mu}+2\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu} \tag{88}$$

and

$$\sqrt{-g}s^{\lambda}_{(G)mn}=\frac{\partial(\sqrt{-g}L_G)}{\partial h^m_{.\mu,\lambda}}h_{n\mu}+\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu} \tag{89}$$

might be interpreted as the spin density of matter field and of gravitational field respectively. But we must indicate in that $\sqrt{-g}s^{.\lambda}_{(M)mn}$, $\sqrt{-g}s^{.\lambda}_{(G)mn}$ lack the invariant character and (87) is not a covariant relation. However if we use (77) to define

$$\begin{aligned}\sqrt{-g}S^{\lambda}_{(M)mn}&=\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{mn}_{..\lambda}}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right)=\frac{1}{2}\frac{\partial(\sqrt{-g}L_M)}{\partial\psi_{,\lambda}}\sigma_{mn}\psi\\&\quad+\frac{\partial(\sqrt{-g}L_M)}{\partial h^m_{.\mu,\lambda}}h_{n\mu}+2\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_M)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right)\end{aligned} \tag{90}$$

and use (78) to define

$$\begin{aligned}\sqrt{-g}S^{\lambda}_{(G)mn}&=\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{mn}_{..\lambda}}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right)\\&=\frac{\partial(\sqrt{-g}L_G)}{\partial h^m_{.\mu,\lambda}}h_{n\mu}+2\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{km}_{..\mu,\lambda}}\Gamma^k_{.n\mu}-\frac{\partial}{\partial x^\mu}\left(\frac{\partial(\sqrt{-g}L_G)}{\partial\Gamma^{mn}_{..\lambda,\mu}}\right)\end{aligned} \tag{91}$$

then we get

$$\sqrt{-g}S^{\lambda}_{(M)mn}+\sqrt{-g}S^{\lambda}_{(G)mn}=0 \tag{92}$$

$$\frac{\partial}{\partial x^\lambda}(\sqrt{-g}S^{\lambda}_{(M)mn}+\sqrt{-g}S^{\lambda}_{(G)mn})=0 \tag{93}$$

Here $\sqrt{-g}S^{.\lambda}_{(M)mn}$, $\sqrt{-g}S^{.\lambda}_{(G)mn}$ and (93) all have the invariant character, hence we will take (92, 93) to be the conservation laws of spin density for the gravitational system with our further generalized Lagrangian densities.

5 Some Special Cases of (10) and (11)

We have indicated that

$$\sqrt{-g(x)}L_M(x)=\sqrt{-g(x)}L_M[\psi(x);\psi_{,\mu}(x);h^i_{.\mu}(x);h^i_{.\mu,\lambda}(x);\Gamma^{ij}_{..\mu}(x)]$$

$$\sqrt{-g(x)}L_M(x)=\sqrt{-g(x)}L_M[\psi(x);\psi_{,\mu}(x);h^i_{.\mu}(x);\Gamma^{ij}_{..\mu}(x);\Gamma^{ij}_{..\mu,\lambda}(x)]$$

$$\sqrt{-g(x)}L_M(x)=\sqrt{-g(x)}L_M[\psi(x);\psi_{,\mu}(x);h^i_{.\mu}(x);\Gamma^{ij}_{..\mu}(x)]$$

are all the special cases of (10), and it is evident that

$$\sqrt{-g(x)}L_G(x)=\sqrt{-g(x)}L_G[h^i_{.\mu}(x);\Gamma^{ij}_{..\mu}(x);\Gamma^{ij}_{..\mu,\lambda}(x)]$$

$$\sqrt{-g(x)}L_G(x)=\sqrt{-g(x)}L_G[h^i_{.\mu}(x);h^i_{.\mu,\lambda}(x)]$$

are all the special cases of (11). With the same method to prove

$$\sqrt{-g(x)}L_M(x) = \sqrt{-g(x)}L_M[\psi(x); \psi_{,\mu}(x); h^i_{.\mu}(x); h^i_{.\mu,\lambda}(x); \Gamma^{ij}_{..\mu}(x); \Gamma^{ij}_{..\mu\lambda}(x)]$$
$$= \sqrt{-g(x)}L^\#_M(x) = \sqrt{-g(x)}L^\#_M[\psi(x); \psi_{|\mu}(x); R^{ij}_{..\mu\nu}(x); T^i_{.\mu\nu}(x); h^i_{.\mu}(x)]$$

we can also prove the following relations:

$$\begin{aligned}\sqrt{-g(x)}L_M(x) &= \sqrt{-g(x)}L_M[\psi(x); \psi_{,\mu}(x); h^i_{.\mu}(x); h^i_{.\mu,\lambda}(x); \Gamma^{ij}_{..\mu}(x)]\\ &= \sqrt{-g(x)}L^\#_M(x)\\ &= \sqrt{-g(x)}L^\#_M[\psi(x); \psi_{|\mu}(x); T^i_{.\mu\nu}(x); h^i_{.\mu}(x)]\end{aligned} \tag{94}$$

$$\begin{aligned}\sqrt{-g(x)}L_M(x) &= \sqrt{-g(x)}L_M[\psi(x); \psi_{,\mu}(x); h^i_{.\mu}(x); \Gamma^{ij}_{..\mu}(x); \Gamma^{ij}_{..\mu,\lambda}(x)]\\ &= \sqrt{-g(x)}L^\#_M(x)\\ &= \sqrt{-g(x)}L^\#_M[\psi(x); \psi_{|\mu}(x); R^{ij}_{..\mu\nu}(x); h^i_{.\mu}(x)]\end{aligned} \tag{95}$$

$$\begin{aligned}\sqrt{-g(x)}L_M(x) &= \sqrt{-g(x)}L_M[\psi(x); \psi_{,\mu}(x); h^i_{.\mu}(x); \Gamma^{ij}_{..\mu}(x)]\\ &= \sqrt{-g(x)}L^\#_M(x) = \sqrt{-g(x)}L^\#_M[\psi(x); \psi_{|\mu}(x); h^i_{.\mu}(x)]\end{aligned} \tag{96}$$

$$\begin{aligned}\sqrt{-g(x)}L_G(x) &= \sqrt{-g(x)}L_G[h^i_{.\mu}(x); \Gamma^{ij}_{..\mu}(x); \Gamma^{ij}_{..\mu\lambda}(x)]\\ &= \sqrt{-g(x)}L^\#_G(x) = \sqrt{-g(x)}L^\#_G[R^{ij}_{..\mu\nu}(x); h^i_{.\mu}(x)]\end{aligned} \tag{97}$$

$$\begin{aligned}\sqrt{-g(x)}L_G(x) &= \sqrt{-g(x)}L_G[h^i_{.\mu}(x); h^i_{.\mu,\lambda}(x)]\\ &= \sqrt{-g(x)}L^\#_G(x) = \sqrt{-g(x)}L^\#_G[T^i_{.\mu\nu}(x); h^i_{.\mu}(x)]\end{aligned} \tag{98}$$

It is not difficult to verify that, for the above special cases of (10) and (11), the conservation laws for a gravitational system all have the same mathematical form:

$$\sqrt{-g}T^{.\lambda}_{(M)\alpha} + \sqrt{-g}T^{.\lambda}_{(G)\alpha} = 0$$
$$\frac{\partial}{\partial x^\lambda}(\sqrt{-g}T^{.\lambda}_{(M)\alpha} + \sqrt{-g}T^{.\lambda}_{(G)\alpha}) = 0$$

and

$$\sqrt{-g}S^{.\lambda}_{(M)mn} + \sqrt{-g}S^{.\lambda}_{(G)mn} = 0$$
$$\frac{\partial}{\partial x^\lambda}(\sqrt{-g}S^{.\lambda}_{(M)mn} + \sqrt{-g}S^{.\lambda}_{(G)mn}) = 0$$

Of course the definitions of energy-momentum tensor density and spin density for different Lagrangian densities are different.

From the discussions in the above sections, we have seen that our further generalized Lagrangian density can be used for describing many theories of gravitation. Their general characters are: the gravitational fields could act on the matter field only through covariant derivative, curvature of space-time, and torsion of space-time; the Lagrangian densities of gravitational field are composed of curvature tensor field and torsion tensor field; the conservation laws for a gravitational system all have the same mathematical form. Their peculiarities are: the concrete forms of Lagrangian densities for matter and gravitational field are different, so the couplings between the gravitational fields and matter field are different; the definitions of energy-momentum tensor density and spin density for different Lagrangian densities are different.

Appendix I. Proof of the Relation (5) for the Space-Time without Torsion

The holonomic connection field $\Gamma^{\mu}_{.\nu\lambda}(x)$ is related to $h^{i}_{.\mu}(x)$ and $\Gamma^{ij}_{..\mu}(x)$ by [10]

$$\Gamma^{\mu}_{.\nu\lambda}(x) = h_i^{.\mu}(x)[h^{i}_{.\nu,\lambda}(x) + \Gamma^{i}_{.j\lambda}(x)h^{j}_{.\nu}(x)] \tag{A1}$$

where $\Gamma^{i}_{.j\lambda}(x) = \eta_{jk}\Gamma^{ik}_{..\lambda}(x)$. In addition, $\Gamma^{ij}_{..\mu}(x) = -\Gamma^{ji}_{..\mu}(x)$. Equation (A1) can be derived from the requirement:

$$g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{.\mu\lambda}g_{\sigma\nu} - \Gamma^{\sigma}_{.\nu\lambda}g_{\sigma\mu} = 0 \tag{A2}$$

This requirement guarantees that lengths and angles are preserved under parallel displacement [7]. The torsion tensor is defined by [13]

$$T^{\mu}_{.\nu\lambda} = \frac{1}{2}(\Gamma^{\mu}_{.\nu\lambda} - \Gamma^{\mu}_{.\lambda\nu}) \tag{A3}$$

There exists the relation [13]:

$$\Gamma^{\mu}_{.\nu\lambda} = \{^{\mu}_{\nu\lambda}\} + T^{..\mu}_{\nu\lambda} - T^{.\mu}_{\lambda.\nu} + T^{\mu}_{.\nu\lambda} \tag{A4}$$

where

$$\{^{\mu}_{\nu\lambda}\} = \frac{1}{2}g^{\mu\sigma}(g_{\sigma\lambda,\nu} + g_{\sigma\nu,\lambda} - g_{\nu\lambda,\sigma}) \tag{A5}$$

is the Christoffel symbol. Equation (A4) can be derived from (A2, A3, A5). In the space-time without torsion, from (A4) it is obviously $\Gamma^{\mu}_{.\nu\lambda} = \{^{\mu}_{\nu\lambda}\}$. In this case the relation (5) can be obtained from (A1, A5).

Appendix II. Some Useful Relations of Differential Geometry

At here we introduce some useful relations of differential geometry which will be used in this paper.

The curvature tensor related to connection $\{^{\mu}_{\nu\lambda}\}$ is defined by [6]

$$\overset{(\{\})}{R}{}^{\sigma}_{.\lambda\mu\nu} = \{^{\sigma}_{\lambda\nu}\}_{,\mu} - \{^{\sigma}_{\lambda\mu}\}_{,\nu} + \{^{\sigma}_{\rho\mu}\}\{^{\rho}_{\lambda\nu}\} - \{^{\sigma}_{\rho\nu}\}\{^{\rho}_{\lambda\mu}\} \tag{A6}$$

Similarly the curvature tensor related to connection $\Gamma^{\mu}_{.\nu\lambda}$ is defined by

$$\overset{(\Gamma)}{R}{}^{\sigma}_{.\lambda\mu\nu} = \Gamma^{\sigma}_{.\lambda\nu,\mu} - \Gamma^{\sigma}_{.\lambda\mu,\nu} + \Gamma^{\sigma}_{.\rho\mu}\Gamma^{\rho}_{.\lambda\nu} - \Gamma^{\sigma}_{.\rho\nu}\Gamma^{\rho}_{.\lambda\mu} \tag{A7}$$

Equation (A4) suggests that, $\overset{(\Gamma)}{R}{}^{\sigma}_{.\lambda\mu\nu} \neq \overset{(\{\})}{R}{}^{\sigma}_{.\lambda\mu\nu}$ for the space-time with torsion; and $\overset{(\Gamma)}{R}{}^{\sigma}_{.\lambda\mu\nu} = \overset{(\{\})}{R}{}^{\sigma}_{.\lambda\mu\nu}$ only for the space-time without torsion. We can also define the curvature tensor related to the frame connection $\Gamma^{ij}_{..\mu}$ [10]:

$$R^{ij}_{..\mu\nu} = \Gamma^{ij}_{..\nu,\mu} - \Gamma^{ij}_{..\mu,\nu} + \Gamma^{i}_{.k\mu}\Gamma^{kj}_{..\nu} - \Gamma^{i}_{.k\nu}\Gamma^{kj}_{..\mu} \tag{A8}$$

Using (A1) it can be verified that $R^{ij}_{..\mu\nu} = \eta^{jk} h^{i}_{.\sigma} h^{.\lambda}_{k} \overset{(\Gamma)}{R}{}^{\sigma}_{.\lambda\mu\nu}$. Using (A1), the following relation for torsion tensor can also be verified:

$$T^{i}_{.\mu\nu}(x) = h^{i}_{.\lambda} T^{\lambda}_{.\mu\nu} = \frac{1}{2}(h^{i}_{.\mu,\nu} - h^{i}_{.\nu,\mu}) + \frac{1}{2}(\Gamma^{i}_{.j\nu} h^{j}_{.\mu} - \Gamma^{i}_{.j\mu} h^{j}_{.\nu}) \tag{A9}$$

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